AN EXAMPLE OF NORMAL LOCAL RING WHICH IS ANALYTICALLY RAMIFIED

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Previously the following question was offered by Zariski [6]:

Is any normal Noetherian local ring analytically irreducible?[1]

In the present note, we will construct a counter-example against the question.

TERMINOLOGY. A ring (integrity domain) means always a commutative ring (integrity domain) with identity. A normal ring is an integrity domain which is integrally closed in its field of quotients. When $\mathfrak{o}$ is an integrity domain, the integral closure of $\mathfrak{o}$ in its field of quotients is called the derived normal ring of $\mathfrak{o}$.

In our treatment, some basic notions and results on general commutative rings and Noetherian local rings are assumed to be well known (see, for example, [5] and one of [1] or [2]). In particular, some results on regular local rings and completions of local rings are used freely (without references). On the other hand, we will make use of an example constructed in [3, § 1] without proof.

§ 1. The construction of an example

Let $\mathbf{k}_0$ be a perfect field of characteristic 2 and let $u_0, v_0, \ldots, u_n, v_n, \ldots$ (infinitely many) be algebraically independent elements over $\mathbf{k}_0$. Set $\mathbf{k} = \mathbf{k}_0(u_0, v_0, \ldots, u_n, v_n, \ldots)$. Further let x and y be indeterminates and set $\mathfrak{r} = \mathbf{k}\{x, y\}$ (formal power series ring), $\mathfrak{o} = \mathbf{k}^2\{x, y\}[\mathbf{k}]$ and $c = \sum_{i=0}^{\infty}(u_i x^i + v_i y^i)$. Then we set $\mathfrak{s} = \mathfrak{o}[c]$.

PROPOSITION. *$\mathfrak{s}$ is a normal Noetherian local ring and the completion of $\mathfrak{s}$ contains non-zero nilpotent elements (that is, $\mathfrak{s}$ is analytically ramified).*

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[1] It was conjectured that the answer is negative by [4] and the present paper answers the conjecture affirmatively.

§ 2. Some preliminary results

LEMMA 1. *The ring* $\mathfrak{o}$ *is a regular local ring with a regular system of parameters* x, y. $\mathfrak{r}$ *is the completion of* $\mathfrak{o}$.

For the proof, see [3, § 1].

LEMMA 2. *An element* $\sum a_{ij}x^iy^j$ $(a_{ij} \in \mathbf{k})$ *is in* $\mathfrak{o}$ *if and only if* $[\mathbf{k}^2(a_{00}, a_{01}, a_{10}, \ldots) : \mathbf{k}^2]$ *is finite.*

Proof. $b = \sum a_{ij}x^iy^j$ is in $\mathfrak{o}$ if and only if b is in $\mathbf{k}^2\{x, y\}[u_0, v_0, \ldots, u_n, v_n]$ for some n. Therefore we see our assertion easily.

LEMMA 3. *Set* $d_n = \sum_{i=0}^{\infty} u_{n+i}x^i$, $e_n = \sum_{i=0}^{\infty} v_{n+i}y^i$ $(n = 0, 1, \ldots)$. *Then* $\mathfrak{t} = \mathfrak{o}[d_0, e_0, \ldots, d_n, e_n, \ldots]$ *is a normal ring.*[2)]

Proof. Let f be any element of the derived normal ring of $\mathfrak{t}$. Since f is in the field of quotients of $\mathfrak{t}$, f is expressed in the form $(p + qd_n + re_n + se_nd_n)/t$ $(p, q, r, s, t \in \mathfrak{o}, t \neq 0)$ (because $\mathfrak{o}[d_0, e_0, \ldots, d_n, e_n] = \mathfrak{o}[d_n, e_n]$ by the construction). Since p, q, r, s and t are in $\mathfrak{o}$, there exists an integer N which is not less than n such that the coefficients of them (as the power series in x and y) are in $\mathbf{k}^2(u_0, v_0, \ldots, u_{N-1}, v_{N-1})$. Then since $d_n = u_n + u_{n+1}x + \ldots + u_{N-1}x^{N-n-1} + x^{N-n}d_N$ and $e_n = v_n + \ldots + v_{N-1}y^{N-n-1} + y^{N-n}e_N$, f is in the derived normal ring of $\mathfrak{o}^*[d_N, e_N]$, where $\mathfrak{o}^* = \mathbf{k}^2\{x, y\}[u_0, v_0, \ldots, u_{N-1}, v_{N-1}]$ (because p, q, r, s and t are in $\mathfrak{o}^*$ by our assumption and because the square of f is in $\mathbf{k}^2\{x, y\} \subseteqq \mathfrak{o}^*$). Since the maximal ideal of $\mathfrak{o}^*$ is generated by x and y, as is easily seen, $\mathfrak{o}^*$ is a (complete) regular local ring. Since the residue class field of $\mathfrak{o}^*$ is represented by $\mathbf{k}^2(u_0, v_0, \ldots, u_{N-1}, v_{N-1})$ and since the leading forms of d_N and e_N are u_N and v_N (respectively), the maximal ideal of $\mathfrak{o}^*[d_N, e_N]$ is generated by x and y. Therefore $\mathfrak{o}^*[d_N, e_N]$ is a regular local ring and is a normal ring. Therefore f is in $\mathfrak{o}^*[d_N, e_N]$ and therefore f is in $\mathfrak{t}$. Thus we see that $\mathfrak{t}$ is normal.

LEMMA 4. $x\mathfrak{S}$ *and* $y\mathfrak{S}$ *are prime ideals.*

Proof. $\mathfrak{S}/x\mathfrak{S}$ is isomorphic to $\mathbf{k}^2\{y\}[\mathbf{k}][e_0]$, which is an integrity domain. Therefore $x\mathfrak{S}$ is a prime ideal. That $y\mathfrak{S}$ is prime follows similarly.

LEMMA 5. *Let* $\mathfrak{S}'$ *be the derived normal ring of* $\mathfrak{S}$ *and let* f *be an element*

2) By virtue of this result, we see easily that $\mathfrak{t}$ is a regular local ring.

of $\mathfrak{S}'$. *If* xyf *is in* $\mathfrak{S}$, *then* f *is in* $\mathfrak{S}$.

Proof. Since $\mathfrak{o}$ is Noetherian and since $\mathfrak{S} = \mathfrak{o}[c]$, $\mathfrak{S}$ is Noetherian. Therefore if f is not in $\mathfrak{S}$, then one of the following must hold (see [5, § 8]): 1) $xy\mathfrak{S}$ has an imbedded prime divisor; 2) there exists at least one minimal prime divisor $\mathfrak{p}$ of $xy\mathfrak{S}$ such that $\mathfrak{S}_{\mathfrak{p}}$ is not normal. Both are impossible because $x\mathfrak{S}$ and $y\mathfrak{S}$ are prime ideals by Lemma 4. Thus we see that f is in $\mathfrak{S}$.

§ 3. Proof of the proposition

As was noted above, $\mathfrak{S}$ is Noetherian. Since $\mathfrak{S}$ is isomorphic to $\mathfrak{o}[X]/g(X)\mathfrak{o}[X]$, where $g(X) = X^2 - c^2$, the completion of $\mathfrak{S}$ is isomorphic to $\mathfrak{r}[X]/g(X)\mathfrak{r}[X]$ (because $\mathfrak{r}$ is the completion of $\mathfrak{o}$). The residue class of $X - c$ is not zero and is nilpotent. Therefore the completion of $\mathfrak{S}$ contains non-zero nilpotent elements. Now we will show that $\mathfrak{S}$ is normal. Let f be any element of $\mathfrak{S}'$. Since $\mathfrak{S}$ is contained in $\mathfrak{t}$ (because $c = d_0 + e_0$) and since $\mathfrak{t}$ is normal by Lemma 3, f is in $\mathfrak{t}$. Therefore f is of the form $p + qd_n + re_n + sd_ne_n$ ($p, q, r, s \in \mathfrak{o}$). Then x^ny^nf is in $\mathfrak{o}[d_0, e_0]$. In order to show that f is in $\mathfrak{S}$, we have only to show that x^ny^nf is in $\mathfrak{S}$ by Lemma 5. Therefore we may assume that $n = 0$. Since f is in the field of quotients of $\mathfrak{S}$, f is of the form $(t + uc)/v$ ($t, u, v \in \mathfrak{o}$). Since $c = d_0 + e_0$, we see that $(t/v) + (u/v)d_0 + (u/v)e_0 = p + qd_0 + re_0 + sd_0e_0$. Since 1, d_0, e_0, d_0e_0 are linearly independent over $\mathfrak{o}$, we have $t/v = p$, $u/v = q(= r, s = 0)$. Therefore $f = p + qc$, which is in $\mathfrak{S}$. Therefore $\mathfrak{S}$ is a normal ring.

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